Solid State Physics IV

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Topology in magnetism; Topological Hall effect

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Announcements.

- Next lecture: Gong sensei
- Problem set (printout and as PDF on website)—deadline Dec. 15.
- Demonstration of 1D-channel conductance included in Lecture 5 lecture note on course website.

First, we repeat a few points from the last lecture.

5.6.2 Fermi-Arc Surface States of a Weyl Semimetal

We consider two Weyl nodes of opposite chirality at $\mathbf{k} = (0, 0, \pm k_z^W)$. Set E_F at the nodes. The Berry curvature acts as an emergent magnetic field in **k**-space, radiating from/into the nodes.

Cut the 3D Brillouin zone (BZ) into 2D slabs of fixed k_z. Each slab behaves as a 2D insulator except exactly at a Weyl node. For $-k_z^W < k_z < k_z^W$ the 2D slice carries Chern number $\nu = 1$ and supports a single chiral edge mode on each 1D edge in *real-space*.

On the *left* and *right* sample surfaces, the 1D surface Fermi "points" occur at different momenta ($\pm k_F$) and have opposite group velocities. Reassembling all k_z-slices yields a *surface* Fermi contour that connects the projections of the two Weyl nodes: the **Fermi arc**. At the nodes, surface states merge with bulk states.

(End of previous lecture.)

5.6.3 Anomalous quantum oscillations in WSM

Reference: A.C. Potter *et al.*, Nat. Commun. **5**, 5161 (2014)

With a magnetic field applied along k_y (so that k_x, k_z no longer label good quantum numbers), electrons are quantized into orbits in a plane perpendicular to the magnetic field. In a Weyl semimetal, electrons at the surface can move along a surface Fermi arc, *tunnel through the bulk*, and complete the orbit on the opposite surface, giving Weyl-orbit quantum oscillations.

6 Topology in magnetism: The magnetic skyrmion

6.1 Winding number of magnetic textures: 1D chain

Considering Ising spins on a 1D chain (up / down degree of freedom only), we only have 'abrupt domain walls', where a spin flips from up to down. Here, the dimension of the chain $d = 1$ is equal to the spin-dimension $d_s = 1$. There is no chirality or handedness associated with this abrupt domain wall.

If however we allow for a rotation of the spin in a two-dimensional plane ($d = 1$, $d_s = 2$), then twisting of the domain wall occurs in two types: left-handed and right-handed twistings are allowed. (this can be considered both for cycloidal or 'Néel-type' domain walls, and for spiral or 'Bloch-type' domain walls). Hence, the possibility to define chiral domain walls depends on the dimensionalities of the lattice and spin.

We aim to define a winding number for spins on the 1D chain by mapping from the infinite 1D line with periodic boundary conditions onto a circle (stereographic projection). The details were illustrated by sketches in the lecture itself, but are not reproduced here. Consider the magnetization (vector, antiparallel to the spin direction) as a map $\mathbf{n} : \mathbb{S}^1 \to \mathbb{S}^1$, which assigns a number $\phi \in [0, 2\pi)$ (tilt angle ϕ of $\mathbf{n}$) to each angle $\alpha \in [0, 2\pi)$ (position on the circle $\mathbb{S}^1$). We map from the circle to the infinite 1D line using a stereographic projection of the type $f : \mathbb{S}^1 \to \mathbb{R}^1 \cup \{\infty\}$, which provides an angle α given a position $x \in (-\infty, \infty]$. We postulate an expression for the winding number in 1D

$$W_1(\mathbf{n}) = \frac{\text{oriented area of } \mathbf{n}}{\text{area of } \mathbb{S}^1} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \left(\frac{\partial \phi}{\partial \alpha}\right) = \frac{\Delta\phi}{2\pi} \tag{1}$$

The latter equality shows that this quantity satisfies the naive expectation for a *winding number*. The object $W_1(\mathbf{n})$ is related to the concept of topological stability, as visualized by the concept of *smoothly unwinding the spin texture*. Speaking to the point, we cannot transfer our spin chain from a state with one winding number to another by smoothly tilting the spins as a whole; if W_1 is changed, the process goes hand in hand with a flip of a single spin with respect to its neighbors.

The above definition of W_1 is suitable for spins on a circle, but a 'more real-world' scenario is as follows: Say we want to parametrize magnetic moments on the 1D chain viz.

$$\mathbf{n} = \begin{pmatrix} \cos\phi(x) \\ \\ \sin\phi(x) \end{pmatrix} \tag{2}$$

with an angle $\phi \in [0, 2\pi)$. Then the quantity

$$W_1(\mathbf{n}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \left(n_x \frac{\partial n_y}{\partial x} - \frac{\partial n_x}{\partial x} n_y\right) \tag{3}$$

is a suitable definition of the winding number, because

$$W_1(\mathbf{n}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \left(\cos\phi(x)\cos\phi(x)\frac{\partial\phi}{\partial x} + \sin\phi(x)\sin\phi(x)\frac{\partial\phi}{\partial x}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \frac{\partial\phi}{\partial x} = \frac{\Delta\phi}{2\pi} \tag{4}$$

as expected for the winding number. This invariant is robust against smooth deformations that keep the mapping continuous.

6.2 Winding number in terms of spin-Berry connection

Reference: P. Bruno *et al.*, Phys. Rev. Lett. **93**, 096806 (2004)

For the $2D$ case, the spin Berry connection is written as

$$a_\mu(\mathbf{r}) \ = \ \frac{\phi_0}{2\pi}\frac{n_x\,\partial_\mu n_y - n_y\,\partial_\mu n_x}{1+n_z}, \qquad \mu \in \{x, y\}, \tag{5}$$

where $\phi_0 = h/e$ is the flux quantum.

We simplify for $d=1$, $d_s=2$, $n_z=0$:

$$a_\mu(\mathbf{r}) = \frac{\phi_0}{2\pi}\,\left(n_x\,\partial_\mu n_y - n_y\,\partial_\mu n_x\right) \tag{6}$$

and the winding number is

$$W_1(\mathbf{n}) = \frac{1}{\phi_0}\int dx\, a_x(x) \tag{7}$$

The Berry connection will appear again later, when we discuss the coupling of moving conduction electrons to the spin texture of localized magnetic moments.

6.3 Winding number of magnetic textures: 2D plane

First consider a two-dimensional plane with spins that can rotate in three dimensions, $d=2$ with $d_s=3$. Depending on the character of spin-twisting, we can smoothly unwind the spin texture or not. First, let us consider an XY vortex where the spins are all arranged in the 2D plane, and form concentric loops around a singularity at $x=y=0$. In the $d_s=3$ case, this XY vortex may be smoothly transferred to a state with zero winding ($\mathbf{n}_i \,//\, \hat{z}$ for all sites i) by tilting the moments towards the $\mathbf{z}$ axis. Hence, the topological winding number in the $d=2$, $d_s=3$ case cannot be simply a vorticity of the type that characterizes an XY vortex.

Instead, the correct definition of the winding number (or skyrmion number, in 2D) is

$$W_{\mathrm{S}}(\mathbf{n}) = \frac{1}{4\pi}\int d^2r\,\mathbf{n}\cdot\left(\frac{\partial\mathbf{n}}{\partial x}\times\frac{\partial\mathbf{n}}{\partial y}\right) \tag{8}$$

We aim to show, in the following, that this object provides an adequate definition of the winding number.

We write $\mathbf{r} = \rho(\cos\alpha, \sin\alpha, 0)^T$ with radius ρ in cylindrical coordinates, which will prove useful in the following. The initial aim is to translate the expression for W to the cylindrical frame. Consider

$$\frac{\partial\mathbf{n}}{\partial x} = \frac{\partial\mathbf{n}}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial\mathbf{n}}{\partial\alpha}\frac{\partial\alpha}{\partial x} \tag{9}$$

and likewise for $\partial\mathbf{n}/\partial y$. Moreover, $\rho = \sqrt{x^2+y^2}$ so that $\partial\rho/\partial x = x/\rho$, $\partial\rho/\partial y = y/\rho$. We also have $\alpha = \arctan y/x$ and hence $\partial\alpha/\partial x = -y/\rho^2$ and $\partial\alpha/\partial y = x/\rho^2$. Taken together,

$$\frac{\partial\mathbf{n}}{\partial x}\times\frac{\partial\mathbf{n}}{\partial y} = \left(\frac{\partial\mathbf{n}}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial\mathbf{n}}{\partial\alpha}\frac{\partial\alpha}{\partial x}\right)\times\left(\frac{\partial\mathbf{n}}{\partial\rho}\frac{\partial\rho}{\partial y} + \frac{\partial\mathbf{n}}{\partial\alpha}\frac{\partial\alpha}{\partial y}\right) = \left(\frac{\partial\mathbf{n}}{\partial\rho}\times\frac{\partial\mathbf{n}}{\partial\alpha}\right)\left(\frac{\partial\rho}{\partial x}\frac{\partial\alpha}{\partial y} - \frac{\partial\alpha}{\partial x}\frac{\partial\rho}{\partial y}\right) \tag{10}$$

and the second part of the final expression simplifies to $1/\rho$ using the expressions in the previous paragraph. As the integration measure transforms as $dxdy \to \rho d\rho d\alpha$ when we move into the cylindrical frame, we arrive at

$$W_{\mathrm{S}}(\mathbf{n}) = \frac{1}{4\pi}\int_0^\infty d\rho \int_0^{2\pi} d\alpha\, \mathbf{n}\cdot\left(\frac{\partial \mathbf{n}}{\partial \rho}\times\frac{\partial \mathbf{n}}{\partial \alpha}\right) \tag{11}$$

For the remainder of this section, assume cylindrical symmetry of our spin vortex for which W_{S} is to be considered, and also demand that at infinite distances $|\mathbf{r}| \to \infty$ the magnetic texture is ferromagnetic (all $\mathbf{n}$ up, i.e. all spins down). The latter condition can be seen as analogous to the demand that the 1D chain should have periodic boundary conditions, see above. Under these conditions, the magnetization field may be written as

$$\mathbf{n}(\mathbf{r}) = \mathbf{n}(\phi, n_z) = \begin{pmatrix} \sqrt{1-n_z^2}\cos\phi \\ \sqrt{1-n_z^2}\sin\phi \\ n_z \end{pmatrix} \tag{12}$$

Given the assumptions, we have $\phi \equiv \phi(\alpha)$ for the moment's tilt angle, and $n_z \equiv n_z(\rho)$ for the z-component of each magnetic moment. We get

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial \rho} &= \frac{\partial \mathbf{n}}{\partial n_z}\frac{\partial n_z}{\partial \rho} = \frac{\partial n_z}{\partial \rho}\begin{pmatrix} -n_z\cos\phi/\sqrt{1-n_z^2} \\ -n_z\sin\phi/\sqrt{1-n_z^2} \\ 1 \end{pmatrix} \\ \frac{\partial \mathbf{n}}{\partial \alpha} &= \frac{\partial \mathbf{n}}{\partial \phi}\frac{\partial \phi}{\partial \alpha} = \frac{\partial \phi}{\partial \alpha}\begin{pmatrix} \sqrt{1-n_z^2}(-\sin\phi) \\ \sqrt{1-n_z^2}(\cos\phi) \\ 0 \end{pmatrix} \end{aligned} \tag{13}$$

and in consequence

$$\mathbf{n}\cdot\left(\frac{\partial \mathbf{n}}{\partial \rho}\times\frac{\partial \mathbf{n}}{\partial \alpha}\right) = -1\cdot\left(\frac{\partial n_z}{\partial \rho}\right)\left(\frac{\partial \phi}{\partial \alpha}\right) \tag{14}$$

so that Eq. (11) simplifies to

$$W_{\mathrm{S}}(\mathbf{n}) = -\frac{1}{4\pi}\int_0^\infty d\rho \int_0^{2\pi} d\alpha \left(\frac{\partial n_z}{\partial \rho}\right)\left(\frac{\partial \phi}{\partial \alpha}\right) = -\frac{\Delta n_z}{4\pi}\,\Delta\phi \tag{15}$$

$\Delta n_z \in \{-2, 0, 2\}$ due to the periodic boundary condition and assuming a maximum of one winding, while

$\Delta\phi = 2\pi N$ with integer N is required by continuity of the magnetization field. Hence, we can distinguish the case $W_\mathrm{S} = 0$ (trivial) and $W_\mathrm{S} = \pm N$ (non-trivial) for this winding number in the cylindrical case.

6.4 Winding number and Berry connection in 2D

Consider again the Berry connection,

$$a_\mu(\mathbf{r}) \;=\; \frac{\Phi_0}{2\pi}\,\frac{n_x\,\partial_\mu n_y - n_y\,\partial_\mu n_x}{1+n_z}, \qquad \mu \in \{x, y\}, \tag{16}$$

where $\phi_0 = h/e$ is the flux quantum. This can be rewritten in the coordinate system of the previous section:

$$a_\mu(\mathbf{r}) \;=\; \frac{\phi_0}{\pi}\,\sin^2\left(\frac{\theta}{2}\right)(\partial_\mu\phi) \tag{17}$$

whereas, the derivation is a bit tedious. The corresponding emergent magnetic field (spin-Berry curvature) is

$$b_z(\mathbf{r}) \;=\; \partial_x a_y - \partial_y a_x \;=\; \frac{\phi_0}{4\pi}\,\hat{\mathbf{n}}\cdot\big(\partial_x\hat{\mathbf{n}}\times\partial_y\hat{\mathbf{n}}\big). \tag{18}$$

These discussions imply that one skyrmion ($W_\mathrm{S} = 1$) carries one flux quantum of emergent magnetic field:

$$\int d^2r\, b_z(\mathbf{r}) = \phi_0\, W_\mathrm{S} \tag{19}$$

Likewise,

$$W_\mathrm{S} = \frac{1}{\phi_0}\int d^2r\,(\nabla\times\mathbf{a})_z \tag{20}$$

6.5 Topological Hall effect

We couple the spin of a moving conduction electron, σ, to the classical spin of localized magnetic moments $\hat{\mathbf{n}}_i$. The corresponding (Kondo) Hamiltonian is

$$i\hbar\partial_t\psi_0 = \left[\frac{(-i\nabla)^2}{2m}\mathbb{1}_2 - J_K\hat{\mathbf{n}}\cdot\sigma\right]\psi_0 \tag{21}$$

where σ is the Pauli 'vector' of Pauli matrices, $\sigma = (\sigma_x, \sigma_y, \sigma_z)^T$, J_K is the Kondo coupling. The spinor wavefunction ψ_0 has spin-up and spin-down components. We take the 'double-exchange limit', $J_K \to \infty$, so that the conduction electron spin follows the local moments adiabatically.

The Hamiltonian is a coupled system of two differential equations. To decouple the two equations, we introduce a local rotation of the coordinate system parallel to $\hat{\mathbf{n}}$ in the form of a unitary transformation (this is the representation of the rotation operation in the Hilbert space of spinor wavefunctions)

$$U(\mathbf{r},t) = \exp\left(-i\frac{\theta}{2}\sigma \cdot \mathbf{\Omega}(\mathbf{r},t)\right) \tag{22}$$

and the rotation axis is

$$\mathbf{\Omega}(\mathbf{r},t) = \frac{\hat{\mathbf{e}}_z \times \hat{\mathbf{n}}}{|\hat{\mathbf{e}}_z \times \hat{\mathbf{n}}|} \tag{23}$$

so that the wavefunction transforms as

$$\psi_0 \to U\psi_0 = \psi = (\psi_\uparrow, \psi_\downarrow)^{\mathrm{T}} \tag{24}$$

In the locally rotated frame, the Hamiltonian is diagonal in the spin component, but we have to pay a 'price': A virtual, or emergent, magnetic field acting on $\psi_\uparrow$, $\psi_\downarrow$:

$$i\hbar\partial_t\psi = \left[\frac{(-i\nabla - q_{\mathrm{em}}\mathbf{a})^2}{2m}\mathbb{1}_2 - J_K\sigma_z + q_{\mathrm{em}}V\right]\psi \tag{25}$$

We have

$$\mathbf{a} = -\frac{i\hbar}{q_{\mathrm{em}}}U^\dagger\nabla U \tag{26}$$

$$V = -\frac{i\hbar}{q_{\mathrm{em}}}U^\dagger\partial_t U \tag{27}$$

where q_{em} was introduced to maintain the conventions from electromagnetic theory. It can be shown in very tedious calculations that this expression for $\mathbf{a}$ reproduces

$$b_z = (\nabla \times \mathbf{a})_z = \frac{\phi_0}{4\pi}\,\hat{\mathbf{n}} \cdot (\partial_x\hat{\mathbf{n}} \times \partial_y\hat{\mathbf{n}}) \tag{28}$$

The emergent field $\mathbf{b}$ acts on charge carriers moving adiabatically through the spin texture of local magnetic moments. It produces an effective Lorentz force on moving charge carriers which can be measured as an additional Hall effect ("topological Hall effect") beyond the ordinary Hall effect (orbital motion in a field) and anomalous Hall effect (Berry phase in $\mathbf{k}$- space discussed for the Weyl semimetal).

In a single-band semiclassical picture,

$$\rho_{yx}^{\mathrm{topo}} \;\propto\; \langle b_z \rangle, \tag{29}$$

with $\langle b_z \rangle$ the real-space average of Eq. (18). (Detailed coefficients depend on spin polarization of electronic bands and so on.)

Order-of-magnitude of the emergent field

For a skyrmion of radius a the typical emergent field scale is

$$|\mathbf{b}| \sim \frac{\phi_0}{a^2} \;\approx\; 10^3 \text{ T} \quad \text{for } a \sim 2 \text{ nm}. \tag{30}$$

where a is the size of a magnetic skyrmion. Thus $|\mathbf{b}|$ can be extremely large when the spin texture has a very short period (nanometric texture).

6.6 Emergent electric field from spin dynamics

Reference: N. Nagaosa, Jpn. J. Appl. Phys. **58**, 120909 (2019)

A time-dependent texture generates an emergent electric field

$$e_i^{\mathrm{em}}(\mathbf{r}, t) \;=\; -\partial_i V - \partial_t a_i = \mp \frac{\hbar}{2q_{\mathrm{em}}}\, \hat{\mathbf{n}} \cdot \big(\partial_i \hat{\mathbf{n}} \times \partial_t \hat{\mathbf{n}}\big) \tag{31}$$

which can be related to a winding number in the space-time domain. This emergent electric field, which is induced by spin dynamics (e.g. from an applied current), is a focus of recent studies in experimental condensed matter physics.