Solid State Physics IV

Chapter 5 October 2025

Berry phase and Hall effect

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Version 1.0 (10/25/2025)

5 Berry Phase and Hall Effect

So far:

- Monopoles of emergent magnetic field (*Berry curvature*).
- Relation of $\mathbf{\Omega}_{n\mathbf{k}}$ to the geometry of parameter space and to the handedness of particle motion (*chiral anomaly*).

Our purpose now:

1. To relate experimental observables such as σ_{xy} to the Berry curvature $\mathbf{\Omega}_{n\mathbf{k}}$ and emergent vector potential $\mathbf{A}_n(\mathbf{k})$.
2. To derive the Kubo formula for the Hall conductivity in linear response theory
3. To derive the TKNN formula for the Hall conductivity (limit of $T = 0$)

$$\sigma_{xy} = \frac{e^2}{h} \int_{\mathrm{BZ}} \frac{d^2k}{2\pi} \left[\partial_{k_x} A_{ny}(\mathbf{k}) - \partial_{k_y} A_{nx}(\mathbf{k})\right] \tag{1}$$

where the Berry connection is $\mathbf{A}_n(\mathbf{k}) = -i \langle u_{n\mathbf{k}} | \nabla_{\mathbf{k}} | u_{n\mathbf{k}} \rangle$ with lattice-periodic Bloch states $|u_{n\mathbf{k}}\rangle$, and to show its relation to the first Chern number in the case of a 2D insulator:

$$\sigma_{xy} = \frac{e^2}{h}\nu, \qquad \nu \in \mathbb{Z} \tag{2}$$

Literature: Naoto Nagaosa *et al.*, Rev. Mod. Phys. **82**, 1959 (2010)

(End of previous lecture)

5.1 Kubo formula

The goal of the derivation is the TKNN formula; to arrive at this result, we start by deriving Kubo's formula for the conductivity tensor.

Consider the effect of the electric field E along the y-direction (!) on a quantum mechanical basis set $|n\rangle$ in first-order quantum-mechanical perturbation theory:

$$|n\rangle_E = |n\rangle + \sum_{m(\neq n)} \frac{\langle m| - qEy |n\rangle}{E_n - E_m} |m\rangle \tag{3}$$

where $-qyE$ is the potential energy of a charge q in the electric field E. The corresponding current is defined as the sum over all particles of the product of charge and velocity (a normalization factor L^2 is necessary to make this observable intensive)

$$\begin{aligned} \langle j_x \rangle_E = \frac{1}{L^2} \sum_n f_{\mathrm{FD}}(E_n(\mathbf{k})) \, {}_E\langle n| \, q v_x \, |n\rangle_E = \frac{1}{L^2} \sum_n f_{\mathrm{FD}}(E_n(\mathbf{k})) \times \\ \times \left[\left(\langle n| + \sum_{m(\neq n)} \frac{\langle m| - qEy |n\rangle}{E_n - E_m} \langle m| \right) q v_x \left(|n\rangle + \sum_{l(\neq n)} \frac{\langle l| - qEy |n\rangle}{E_n - E_l} |l\rangle \right) \right] \end{aligned} \tag{4}$$

with f_{FD} the Fermi-Dirac distribution function and $E_n(\mathbf{k})$ the spectrum of the Hamiltonian. We focus on terms which are first order in the perturbing electric field so that

$$\begin{aligned} \langle j_x \rangle_E = \langle j_x \rangle_{E=0} + \frac{1}{L^2} \sum_{n \neq m} f_{\mathrm{FD}}(E_n(\mathbf{k})) \times \\ \left[\frac{\langle n| - qEy |m\rangle \langle m| q v_x |n\rangle}{E_n - E_m} + \frac{\langle n| q v_x |m\rangle \langle m| - qEy |n\rangle}{E_n - E_m} \right] \end{aligned} \tag{5}$$

Recall the Heisenberg equation for operators, $d\hat{\mathcal{O}}/dt = (1/i\hbar)[\hat{\mathcal{O}}, \hat{\mathcal{H}}] + \partial\hat{\mathcal{O}}/\partial t\Big|_{\hat{\mathcal{H}}}$. For the velocity operator, this implies $\hat{\mathbf{v}} = \dot{\mathbf{x}} = (1/i\hbar)[\hat{\mathbf{x}}, \hat{\mathcal{H}}]$ so that

$$\langle m|\, \mathbf{v}\, |n\rangle = \frac{1}{i\hbar} \langle m|\, \hat{\mathbf{x}}\hat{\mathcal{H}} - \hat{\mathcal{H}}\hat{\mathbf{x}}\, |n\rangle = \frac{1}{i\hbar}(E_n - E_m) \langle m|\, \hat{\mathbf{x}}\, |n\rangle \tag{6}$$

For the longitudinal current and the Hall conductivity, this yields

$$\langle j_x\rangle_E = \langle j_x\rangle_{E=0} - \frac{i\hbar E q^2}{L^2} \sum_{n\neq m} \left[\frac{\langle n|\, v_x\, |m\rangle \langle m|\, v_y\, |n\rangle}{(E_n - E_m)^2} + \frac{\langle n|\, v_y\, |m\rangle \langle m|\, v_x\, |n\rangle}{-(E_n - E_m)^2} \right] \tag{7}$$

$$\sigma_{xy} = \frac{1}{E}\left[\langle j_x\rangle_E - \langle j_x\rangle_{E=0}\right] = -\frac{i\hbar q^2}{L^2} \sum_{n\neq m} \left[\frac{\langle n|\, v_x\, |m\rangle \langle m|\, v_y\, |n\rangle}{(E_n - E_m)^2} - \frac{\langle n|\, v_y\, |m\rangle \langle m|\, v_x\, |n\rangle}{(E_n - E_m)^2} \right] \tag{8}$$

and this already demonstrates Onsager's relation for the Hall conductivity, $\sigma_{xy} = -\sigma_{yx}$.

5.2 Bloch Hamiltonian for band electrons

We specialize to the case of Bloch wavefunctions in a periodic potential

$$|\psi_n(\mathbf{k})\rangle = e^{i\mathbf{k}\cdot\mathbf{x}}\, |u_{n\mathbf{k}}\rangle \tag{9}$$

where the $|u_{n\mathbf{k}}(\mathbf{x})\rangle$ are a basis set of lattice-periodic functions, $|u_{n\mathbf{k}}(\mathbf{x}+\mathbf{T})\rangle = |u_{n\mathbf{k}}(\mathbf{x})\rangle$ and $\mathbf{T} = \sum_i n_i \mathbf{a}_i$ with $\mathbf{a}_i$ enumerating all basis vectors of the lattice, and $n_i \in \mathbb{N}$. Further, $\mathbf{k}$ is the crystal momentum – defined only up to a reciprocal lattice vector $\mathbf{G}$ –, n is the band index and $\exp(i\mathbf{k}\cdot\mathbf{x})$ is a plane wave.

The effect of the (full) Hamiltonian on these wavefunctions is

$$\hat{\mathcal{H}}\, |\psi_{n\mathbf{k}}\rangle = E_{n\mathbf{k}}\, |\psi_{n\mathbf{k}}\rangle \tag{10}$$

$$\hat{\mathcal{H}}_{\mathbf{k}}\, |u_{n\mathbf{k}}\rangle = E_{n\mathbf{k}}\, |u_{n\mathbf{k}}\rangle \tag{11}$$

This can be derived from $\hat{\mathcal{H}} = (1/2m)\left(-i\hbar\nabla\right)^2 + A_0(\mathbf{x})$, from which it follows

$$\hat{\mathcal{H}}\, |\psi_{n\mathbf{k}}\rangle = \left[\frac{\hbar^2}{2m}(-i\nabla)^2 + A_0(\mathbf{x})\right] e^{i\mathbf{k}\cdot\mathbf{x}}\, |u_{n\mathbf{k}}(\mathbf{x})\rangle = e^{i\mathbf{k}\cdot\mathbf{x}} \left[\frac{\hbar^2}{2m}(-i\nabla + \mathbf{k})^2 + A_0(\mathbf{x})\right] |u_{n\mathbf{k}}(\mathbf{x})\rangle \tag{12}$$

whereas the term in brackets is $\hat{\mathcal{H}}_{\mathbf{k}}$, i.e., we have $\hat{\mathcal{H}}_{\mathbf{k}} = (1/2m)\left(-i\hbar\nabla + \hbar\mathbf{k}\right)^2 + A_0(\mathbf{x})$. We can also write $\hat{\mathcal{H}}\, |\psi_{n\mathbf{k}}\rangle = e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\mathcal{H}}_{\mathbf{k}}\, |u_{n\mathbf{k}}\rangle$.

5.3 TKNN formula for the Hall conductivity

Regarding the velocity operator, it can be expressed as $\hat{\mathbf{v}} = (1/i\hbar)[\hat{\mathbf{x}}, \hat{\mathcal{H}}]$ or $\hat{\mathbf{v}} = (1/i\hbar)[\hat{\mathbf{x}}, \hat{\mathcal{H}}_{\mathbf{k}}]$, depending on whether we are working with the $|\psi_{n\mathbf{k}}\rangle$ or $|u_{n\mathbf{k}}\rangle$ basis set. In the latter case, we write explicitly

$$\begin{aligned}\langle u_{n\mathbf{k}}|\,\hat{\mathbf{v}}\,|u_{m\mathbf{k}}\rangle &= \frac{1}{i\hbar}\left[\langle u_{n\mathbf{k}}|\, i\nabla_{\mathbf{k}}\left(E_{m\mathbf{k}}\,|u_{m\mathbf{k}}\rangle\right) - \langle u_{n\mathbf{k}}|\, E_{n\mathbf{k}} i\nabla_{\mathbf{k}}\,|u_{m\mathbf{k}}\rangle\right] \\ &= \frac{1}{\hbar}\frac{\partial E_{n\mathbf{k}}}{\partial \mathbf{k}}\delta_{nm} + \frac{1}{\hbar}\left(E_{m\mathbf{k}} - E_{n\mathbf{k}}\right)\left\langle u_{n\mathbf{k}} \middle| \frac{\partial u_{n\mathbf{k}}}{\partial \mathbf{k}}\right\rangle \end{aligned} \tag{13}$$

Along these lines and using $\mathbb{1} = \sum_m |m\rangle\langle m|$ as well as $0 = \nabla_{\mathbf{k}}\langle u_{n\mathbf{k}}|\, u_{m\mathbf{k}}\rangle = \langle \nabla_{\mathbf{k}} u_{n\mathbf{k}}|\, u_{m\mathbf{k}}\rangle + \langle u_{n\mathbf{k}}|\, \nabla_{\mathbf{k}} u_{m\mathbf{k}}\rangle$ for $n \neq m$ *and* $n = m$, the Hall conductivity is rewritten as

$$\begin{aligned}\sigma_{xy} &= -\frac{iq^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n\neq m} f_{\mathrm{FD}}(E_{n\mathbf{k}})\left[\left\langle \frac{\partial u_{n\mathbf{k}}}{\partial k_x}\middle| u_{m\mathbf{k}}\right\rangle\left\langle u_{m\mathbf{k}}\middle|\frac{\partial u_{n\mathbf{k}}}{\partial k_y}\right\rangle - \left\langle \frac{\partial u_{n\mathbf{k}}}{\partial k_y}\middle| u_{m\mathbf{k}}\right\rangle\left\langle u_{m\mathbf{k}}\middle|\frac{\partial u_{n\mathbf{k}}}{\partial k_x}\right\rangle\right] \\ &= -\frac{iq^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n} f_{\mathrm{FD}}(E_{n\mathbf{k}})\left[\left\langle \frac{\partial u_{n\mathbf{k}}}{\partial k_x}\middle| \frac{\partial u_{n\mathbf{k}}}{\partial k_y}\right\rangle - \left\langle \frac{\partial u_{n\mathbf{k}}}{\partial k_y}\middle| \frac{\partial u_{n\mathbf{k}}}{\partial k_x}\right\rangle\right] \\ &= -\frac{iq^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n} f_{\mathrm{FD}}(E_{n\mathbf{k}})\left[\frac{\partial}{\partial k_x}\left\langle u_{n\mathbf{k}}\middle|\frac{\partial u_{n\mathbf{k}}}{\partial k_y}\right\rangle - \frac{\partial}{\partial k_y}\left\langle u_{n\mathbf{k}}\middle|\frac{\partial u_{n\mathbf{k}}}{\partial k_x}\right\rangle\right] \\ &= -\frac{iq^2}{\hbar L^2}\sum_{\mathbf{k}}\sum_{n} f_{\mathrm{FD}}(E_{n\mathbf{k}})\left[\frac{\partial A_{ny}(\mathbf{k})}{\partial k_x} - \frac{\partial A_{nx}(\mathbf{k})}{\partial k_y}\right] \end{aligned} \tag{14}$$

where the Berry connection $\mathbf{A}_n(\mathbf{k}) = -i\,\langle u_{n\mathbf{k}}|\,\nabla_{\mathbf{k}}\,|u_{n\mathbf{k}}\rangle$ was introduced for Bloch states. As the Berry curvature is $\mathbf{\Omega}_n(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathbf{A}_n(\mathbf{k})$, this is the same as

$$\begin{aligned}\sigma_{xy} &= \frac{q^2}{\hbar L^2}\sum_{n}\sum_{\mathbf{k}} f_{\mathrm{FD}}(E_{n\mathbf{k}})\,\Omega_{nz}(\mathbf{k}) \\ &= \frac{q^2}{h}\sum_{n}\int_{\mathrm{BZ}}\frac{d^2\mathbf{k}}{(2\pi)^2} f_{\mathrm{FD}}(E_{n\mathbf{k}})\,\Omega_{nz}(\mathbf{k})\end{aligned} \tag{15}$$

where the second equality is for the 2D problem only. For either 2D or 3D, the result means that the Hall effect in the x-y plane is simply the sum of the Berry curvature's z-component over all occupied states. The present version of the TKNN formula is still valid for either an insulator or a metal.

5.4 Hall effect and monopole density in two dimensions

Purpose here: Relationship between the Hall conductivity σ_{xy} and monopoles, especially for the case of a 2D insulator.

Case without monopoles in the 2D BZ

Considering the two-dimensional case, the area integral over the Brillouin zone can be replaced with (Stokes' theorem) a line integral of $\mathbf{A}_n(\mathbf{k})$ around the whole edge of the BZ:

$$\sigma_{xy} = \frac{1}{2\pi}\frac{q^2}{h}\sum_n \oint_{\mathcal{S}(\mathrm{BZ})} d\mathbf{l}_{\mathbf{k}} \cdot \mathbf{A}_n(\mathbf{k}) \tag{16}$$

where $d\mathbf{k}$ is a unit vector parallel to the surface line $\mathcal{S}(BZ)$ of the Brillouin zone, along which we are integrating. Due to periodicity of the lattice, the left and right or top and bottom boundaries of the BZ must be associated with the same values of $\mathbf{A}_n$, but the integration direction is opposite for each pair of edges. Hence, the line integral must vanish in the 2D case *if there is a single vector potential without singularities describing the Berry phase effect in the whole BZ*: $\sigma_{xy} = 0$

5.5 Anomalous Hall Effect of a 3D Weyl Semimetal

Reference: A. Burkov, *Phys. Rev. Lett.* **113**, 187202 (2014)

We consider the case of two Weyl fermions with opposite chiralities at positions $(0, 0, \pm k_z^W)$ in the Brillouin zone (BZ). We set the Fermi energy E_F to be at the position of the Weyl points (i.e., the Fermi surface of the Weyl points is of zero measure). The Berry curvature $\mathbf{\Omega}_{n\mathbf{k}}$ acts as an emergent magnetic field in momentum space, which points away from / towards the Weyl point(s). We cut the 3D Brillouin zone into 2D slices at constant k_z values. For each slice, the system behaves as an effective 2D insulator, as long as the slice does not exactly cut through the Weyl point itself.

The TKNN formula for each insulating slice is:

$$\sigma_{xy}(k_z) = \frac{e^2}{h}\int \frac{d^2k}{(2\pi)^2}\,\Omega_n^z(\mathbf{k}), \tag{17}$$

where Ω_n^z is the z-component of the Berry curvature.

1. For $-k_z^W < k_z < k_z^W$: the net Berry flux through the 2D slice is finite $\Rightarrow$ contributes a finite amount to σ_{xy}.

2. For $k_z > k_z^W$ or $k_z < -k_z^W$: there is no net flux through the slice, because the emergent magnetic field $\mathbf{\Omega}_{n\mathbf{k}}$ bends in the shape of a dipole around the boundary of the BZ. Therefore, these slices make no contribution to the bulk σ_{xy}.

Integrating the Hall conductivity over k_z gives:

$$\sigma_{xy} = \frac{e^2}{h}\frac{\Delta k_z}{\Delta k_z^{\mathrm{BZ}}} \cdot \frac{1}{c}, \tag{18}$$

where Δk_z, Δk_z^{BZ}, and c are the momentum-space separation between the two Weyl nodes, the dimension of the BZ along the z axis, and the lattice constant along the z-axis, respectively. Note that σ_{xy} in 2D has the same dimension as e^2/h, whereas in 3D we have to add c^{-1} to satisfy the dimensional analysis.

Hence, the anomalous Hall effect depends directly on the distance between Weyl nodes (when properly taking care of their alignment relative to the $k_x = k_y = 0$ line).

5.6 Bulk–Boundary Correspondence: Fermi-Arc Surface States

5.6.1 Laughlin's Argument for a 2D insulator

Purpose: Discuss a simple case of bulk-boundary correspondence, considering an insulator in two dimensions (2D).

Take a rectangular plate of a 2D insulator and wrap it into a cylinder so that the y-axis is along the tube and the x-axis is periodic, with cylinder circumference L_x. We then insert a magnetic field adiabatically slow into the center of the cylinder and an electric field E_x is created tangentially to the cylinder by Faraday's law of induction:

$$E_x = -\dot{\Phi}_{\text{flux}}/L_x \tag{19}$$

where Φ_{flux} is the magnetic flux. As our system is an insulator, we have $\sigma_{xx} = \sigma_{yy} = 0$. However, the Hall conductivity σ_{xy} can be finite, if the insulator is nontrivial, and we here examine the implications of this.

We have

$$j_y = \sigma_{xy} E_x \tag{20}$$

parallel to the axis of the cylinder. As the system is an insulator, the charge that "flows" along the cylinder cannot be understood in the semiclassical way as a motion of delocalized wavepackets throughout the entire sample. Instead, it should be understood quantum mechanically as a displacement of the center of wavefunctions within each unit cell, but a detailed discussion is not attempted here (c.f. Thouless pump).

Let it suffice to say that the charge is moved from the $-y$ edge of the cylinder to the $+y$ edge, if σ_{xy} is finite. Being a bulk insulator, the bulk of the cylinder does not have any place to put the charges; this means, there must be a gapless region at the edge that allows us to store the 'pumped' charge in unoccupied states. This is Laughlin's argument that there must be gapless edge states at the boundary of an insulator with nonzero Hall conductivity.

Let us further understand the logic for why the Hall conductivity must be quantized by calculating the pumped charge in the time interval $[0, T]$, where the flux is increased from 0 to $\phi_0 = h/e$, the flux quantum:

$$Q_{\text{pumped}} = \int_0^T I_y dt = \int_0^T L_x j_y dt = \int_0^T L_x \left(-\frac{d\Phi_{\text{flux}}}{dt} \cdot \frac{1}{L_x} \right) \sigma_{yx} dt = \sigma_{xy} \int_0^{\phi_0} d\Phi_{\text{flux}} = \frac{h}{e} \sigma_{xy} \tag{21}$$

But the charge Q_{pumped} must be an integer multiple of e, the fundamental charge, so that

$$\sigma_{xy} = \frac{e^2}{h} \nu \tag{22}$$

with an integer ν. This means that the Hall conductivity of an insulator in 2D must be quantized (or zero) at low temperatures. It is reasonable that the smallest pumping cycle, ϕ_0, transfers exactly one electron between the sides of the cylinder per edge mode (per available state at the edge), so that ν is also the number of edge modes.

To justify this conclusion, a quantum mechanical argument is important: the insertion of a flux ϕ_0 into the center of the cylinder will return the quantum system into an eigenstate, as in the Aharonov-Bohm effect – although, this is not the ground state if charges have been pumped between the sides of the cylinder. As eigenstates are discretely spaced in energy, the amount of charge is also discrete and the Hall conductivity becomes quantized.

Note: In principle, there is another, equivalent way to discuss the Laughlin argument in terms of a (flat) annular disk, i.e., a Hall device in Corbino geometry.

5.7 Fermi arcs on the surface of Weyl semimetals

Let these be located at $\pm k_z^W$ on the k_z axis. As we are dealing with (semi-) metallic materials here, it is important to realize that surface states are well defined only in regions of the two-dimensional surface Brillouin zone, where there is no corresponding bulk density of states. If there is, surface and bulk states merge into each other. Hence, we consider the notion of a $\mathbf{k}$-dependent band gap, i.e. a band gap which vanishes at the Weyl points but is nonzero everywhere else (in this simplified band structure).

Slicing the Brillouin zone into two-dimensional (infinite) slabs with $k_z^W = \text{const.}$, we see that each slab with $-k_z^W < k_{z,\text{slab}} < k_z^W$ is exposed to a quantized flux of Berry curvature, corresponding to the Chern number ν of the 2D slab. As there are only two Weyl points, each slab with $\nu = 1$ supports one chiral surface mode on its edge in real space, when transitioning from (spatially) infinite to finite slabs.

Consider the 'top' and 'bottom' surfaces where $y = \text{const.}$ and the surface Brillouin zones are spanned by k_x, k_z. Each of these surface Brillouin zones may be cut into one-dimensional slices corresponding to the 2D slabs introduced in the previous paragraph. For each 1D slice, the Fermi surface is a point with $E = +\hbar v k$ on the upper, and $E = -\hbar v k$ on the lower surface, due to the chiral nature of the surface modes on 2D slabs. Reassembling the 2D surface Brillouin zone from 1D slices, we are left with a line of surface Fermi points spanning from one Weyl point to the other, yielding the Fermi arc. Key points here are that

- The surface Fermi ring is essentially cut in half, with the top surface hosting one half, and the bottom surface hosting the other; a consequence of the chiral nature of surface modes driven by the Chern number on each slab.
- Fermi arcs appear only on surfaces where 'paired' Weyl points, i.e. Weyl points connected by a Fermi arc, do not project onto the same point on the surface Brillouin zone.
- At the Weyl points, the surface modes merge with the bulk states (as projected onto the surface) and electrons can transition from surface to bulk.
- While the presence of Fermi arcs is dictated by topology, their precise shape on the surface Brillouin zone, and even their connectivity in cases with more than two Weyl points, depends on details of the

Hamiltonian.

5.8 Appendix: Conductance of a one-dimensional channel

Consider a 1D channel as a constricted piece of semiconductor connected to two reservoirs (left and right) with chemical potential μ_1 and μ_2, respectively. The positive x-direction is from left to right. The Fermi surface of the 1D channel consists of two points with $k = \pm k_F$ (at zero temperature). If we assume a simple quadratic band dispersion $\varepsilon \sim \hbar^2 k_x^2/(2m)$, it is easy to see that $+k_F$ corresponds to a right-mover ($v = \hbar^{-1}\partial E/\partial k_x > 0$ and $-k_F$ to a left-mover. At low T, scattering between these two points requires a large momentum transfer, which cannot be supplied by the low-lying excitations in the system (e.g. phonons). Therefore, the right- and left-movers are essentially decoupled systems, which may maintain different chemical potentials 'without speaking to each other'.

Suppose a particle enters the constricted channel from reservoir 1. Because there is no scattering, it will maintain its chemical potential at μ_1 until hitting the boundary to reservoir 2, where it suddenly drops to lower energy - through dissipative processes such as scattering, which we do not discuss in detail here. Likewise, a particle entering the channel from reservoir 2 will stay at chemical potential μ_2 until reaching the boundary to reservoir 1. This is referred to as *ballistic motion in constricted channel*. In other words, the transport in the channel is dissipationless.

We calculate the total current through the channel as a sum of left- and right-movers. For right-movers,

$$I_R = j_R = q \int_{-\infty}^{\infty} dE g_{\mathrm{1D}}(E) f_{\mathrm{FD}}(E) \cdot v_x \tag{23}$$

The current is the product of the particle's charge, the number of occupied states, and the mean velocity. f_{FD} is the Fermi-Dirac distribution function and the density of states for a one-dimensional channel with the above-mentioned quadratic dispersion, as well as the velocity give the current

$$g_{1\mathrm{D}} = \frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} E^{-1/2} \tag{24}$$

$$v_x = \left(\frac{2E}{m}\right)^{1/2} \tag{25}$$

$$I_R = \frac{q}{2\pi\hbar}\int_{-\infty}^{\infty} dE f_{\mathrm{FD}}(E) \tag{26}$$

Summing up right and left moving currents,

$$I_{tot} = I_R - I_L = \frac{q}{h}\int_{-\infty}^{\infty} dE\left[f_{\mathrm{FD}}(E-\mu_1) - f_{\mathrm{FD}}(E-\mu_2)\right] = \frac{q}{h}\cdot(\mu_1-\mu_2) \tag{27}$$

where the difference of potential energy between the two reservoirs is $q\Delta A_0 = (\mu_1 - \mu_2)$. Therefore, the 1D resistance is $R_{1\mathrm{D}} = \Delta A_0/I = h/q^2$ for a single 1D channel.

In the case of the quantum Hall effect, the chiral edge mode does not allow 'right-movers' and 'left-movers' on the same edge; instead, the two species are separated on different edges, making it even harder to scatter between the two types – not only a large momentum change, but even a change of the spatial position of the wavefunction across the sample would be necessary to transition from $-k_{\mathrm{F}}$ to $+k_{\mathrm{F}}$. The quantization of the one-dimensional semiconductor channel, realized in a device, was indeed observed later than the quantum Hall effect itself, and exclusively at ultra-low temperatures where scattering is suppressed. It can be argued – although this depends sensitively on system parameters – that quantization of the Hall effect in the two-dimensional electron gas is more 'robust' than quantization of a one-dimensional semiconducting channel.

Note: As compared to the simple 1D channel, the edge channel of the Quantum Hall system forms a loop. The reason why, unlike for the 1D channel, no charge is transported through the chiral edge between the reservoirs ($\sigma_{xx} = 0$) while the Hall conductivity σ_{xy} is quantized can be argued on this basis using the Landauer-Buettiker formalism of circuitry (not treated here).