Condensed Matter Physics IV

Chapter 2 October 2025

Recap: Topology in Quantum Mechanics, Aharonov-Bohm, Dirac monopole

The University of Tokyo

Max Hirschberger

hirschberger@ap.t.u-tokyo.ac.jp

http://www.qpec.t.u-tokyo.ac.jp/hirschberger/

Version 1.0 (10/23/2025)

We continue the discussion of Dirac's magnetic monopole.

Our motivation is:

1. Application of gauge transformation to a physical problem

2. Quantization of a physical quantity, here the electric charge of an electron or the magnetic monopole charge of a magnetic monopole, due to single-valued nature of the wavefunction

3. Example of a topological charge without introducing the emergent magnetic field / Berry phase formalism; here we can use entirely the (well-known) electromagnetic fields.

Recap from the previous lecture:

Consider the enhanced symmetry that would be introduced to Maxwell's classical electrodynamics by the presence of a new elementary particle, the magnetic monopole or magnetic charge density ρ_m in vacuum

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \rho_m & \nabla \cdot \mathbf{E} &= \rho_e/\varepsilon_0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{j}_m & \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}_e \end{aligned} \tag{1}$$

where $c = 1/\sqrt{\mu_0 \varepsilon_0}$, μ_0, and ε_0 are the speed of light, the magnetic permeability of vacuum, and the electric permittivity of vacuum, respectively. We also have the continuity equations for the charges, $\partial\rho_e/\partial t + \nabla \cdot \mathbf{j}_e = 0$ and $\partial\rho_m/\partial t + \nabla \cdot \mathbf{j}_m = 0$.

The modified Lorentz force:

The currents $\mathbf{j}_e$ and $\mathbf{j}_m$ are the electric and magnetic current densities. A modified Lorentz force can be derived as

$$\mathbf{F} = q_e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) + \frac{q_m}{\mu_0}\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right), \tag{2}$$

where the symmetry of the dependence on $q = q_e$ and q_m is quite apparent. The magnetic charge is measured in Webers, $[\mathrm{Wb}] = \mathrm{kg\,m^2/A\,s^2}$.

2.4.1 Vector potential for motion in field of magnetic monopole

Contrary to the familiar case of classical electrodynamics, $\nabla \cdot \mathbf{B} = 0$ is now violated due to the presence of magnetic monopoles. While the electromagnetic vector potential $\mathbf{A}$ is still related to the observable magnetic field as $\mathbf{B} = \nabla \times \mathbf{A}$, the mathematical identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ breaks down at singular locations $\mathbf{x}_i$, i.e. at the locations of the monopoles.

Purpose / summary: We are interested in the stationary behavior of a QM particle on the two-dimensional surface of a sphere of radius R around a magnetic monopole. We show that the point-like singularity of $\mathbf{B}$ at the center of the sphere requires that no unique function $\mathbf{A}(\mathbf{x})$ can be defined on the entire spherical surface; instead, we must define $\mathbf{A}$ in at least two separate, but overlapping patches. Drawing an analogy to the previous section: even if the QM particle does not have any probability density at the location of the monopole, its quantum-phase properties are affected by the monopole through geometrical constraints on the vector potential $\mathbf{A}$.

First, we choose an ansatz for the magnetic field of a monopole in analogy to the electric field of a point charge,

$$\mathbf{B} = \frac{q_m}{4\pi r^2} \frac{\mathbf{r}}{r} \tag{3}$$

where $\mathbf{r}/r = \hat{e}_r$ in spherical coordinates. This gives, according to Gauss' theorem with a spherical surface $\mathcal{S}$ of radius R surrounding the point charge,

$$\int_{\mathcal{S}} dr^2 \, \mathbf{B} \cdot \hat{n} = q_m \tag{4}$$

A possible ansatz for the vector potential is

$$\mathbf{A}^{\mathrm{N}}(\mathbf{x}) = \frac{q_m}{4\pi} \frac{1 - \cos\theta}{r \sin\theta} \, \hat{e}_\varphi \tag{5}$$

which is tangential to the spherical surface and diverges at $\theta = \pi$. In spherical coordinates (r, θ, φ), the curl operation is

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta} \left(\frac{\partial}{\partial\theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial\varphi} \right) \hat{e}_r + \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(r A_\varphi) \right) \hat{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta} \right) \hat{e}_\varphi \tag{6}$$

But our expression only includes A_φ and rA_φ is independent of r, so that

$$\frac{4\pi}{q_m}\nabla\times\mathbf{A}^{\mathrm{N}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1-\cos\theta}{r}\right)\hat{e}_r = \frac{1}{r^2}\hat{e}_r \tag{7}$$

as expected for the monopole field. At the south pole $\theta=\pi$ however, the vector potential breaks down and the magnetic field cannot be well defined from $\mathbf{A}^{\mathrm{N}}$ (although $\mathbf{B}$ does not diverge on the spherical surface at radius R). Likewise, it is possible to define a south-pole potential

$$\mathbf{A}^{\mathrm{S}}(\mathbf{x}) = \frac{q_m}{4\pi}\left(\frac{-1-\cos\theta}{r\sin\theta}\right)\hat{e}_\varphi \tag{8}$$

which diverges at the north pole, i.e., at $\theta=0$.

These two vector potentials result in the same magnetic field $\mathbf{B}$, so they must be connected by a gauge transformation

$$\mathbf{A}^{\mathrm{N}} - \mathbf{A}^{\mathrm{S}} = \frac{q_m}{2\pi}\frac{1}{r\sin\theta}\hat{e}_\varphi \overset{!}{=} \nabla\Lambda(\mathbf{x}) \tag{9}$$

with $\Lambda = (q_m/2\pi)\varphi$ and φ is the scalar spherical coordinate related to $\hat{e}_\varphi$. It can be confirmed that

$$\nabla\Lambda(\mathbf{x}) = \hat{e}_r\frac{\partial\Lambda}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial\Lambda}{\partial\theta} + \hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial\varphi} = \frac{q_m}{2\pi}\cdot\frac{1}{r\sin\theta} \tag{10}$$

There is thus no single vector potential that is non-divergent everywhere on the sphere around the monopole. However, we can 'cover the sphere in patches'.

The situation is quite different from an electric monopole, which has a well-defined scalar potential A_0, despite the apparent 'symmetry' of the modified Maxwell equations.

2.4.2 Quantization of electric and magnetic charges

Say that ψ corresponds to a solution of the static Schroedinger equation for a particle with electric charge q under the influence of the magnetic monopole of charge q_m:

$$\mathcal{H}\psi = E\psi \quad\rightarrow\quad \frac{1}{2m}\left(-i\hbar\nabla - q\mathbf{A}\right)^2\psi = E\psi \tag{11}$$

We use $\mathbf{A}^{\mathrm{N}}$ at $0\le\theta\le\pi-\varepsilon$ and $\mathbf{A}^{\mathrm{S}}$ for $\varepsilon\le\theta\le\pi$ with a finite $\varepsilon>0$. In the overlap region, either one of them is suitable. According to the gauge transformation between the vector potentials, discussed above, the wavefunctions transform as

$$\psi \rightarrow \exp\left[iq\Lambda(\mathbf{x})/\hbar\right]\psi \tag{12}$$

so that the wavefunctions in the two patches are related by

$$\psi^{\mathrm{N}} = \exp\left[+\frac{iq}{\hbar}\left(\frac{q_m}{2\pi}\right)\phi\right]\psi^{\mathrm{S}} \tag{13}$$

Considering a loop around the equator, we demand that the wavefunction must return onto itself at $\Delta\varphi = 2\pi N_0$, where $N_0 \in \mathbb{Z}$ (in a later section we derive the same result for bosonic wavefunctions without making this assumption). Hence,

$$\frac{q}{\hbar}\frac{q_m}{2\pi} \overset{!}{=} N_0 \quad \rightarrow \quad q_m q = hN_0 \tag{14}$$

and the product of the two charges is quantized. We note that, for the product to be quantized, neither $q = q_e$ nor q_m can take on continuous values.

Comment 1: The quantization of the electric and magnetic charges also implies the quantization of magnetic flux. The total magnetic flux through the surface of our sphere (radius R) is

$$\int d^2r\,\mathbf{B}\cdot\hat{n} = \int d^3\mathbf{r}\,\nabla\cdot\mathbf{B} = q_m = \frac{hN_0}{q} = \frac{h}{q}N_0 = \phi_0 N_0 \tag{15}$$

where ϕ_0 is the flux quantum for a single particle (not superconducting flux quantum). The first equality arises from the divergence theorem, and the second equality to the modified Maxwell equations.

Comment 2: If there is no magnetic monopole charge, $q_m = 0$, then $A^{\mathrm{S}} = A^{\mathrm{N}}$ and the argument breaks down. We need the monopole at the center to use the argument about the gauge transformation.

3 Berry phase and Berry curvature (nondegenerate case)

3.1 Definition of the Berry curvature

We consider a parameter space $\mathbf{R}(t)$, where t is an intrinsic parameter which allows us to traverse a path in the parameter space. For the purpose of the present discussion, $\mathbf{R}$ is a position in a three-dimensional vector space. For parameter-dependent Hamiltonian $\hat{\mathcal{H}}(\mathbf{R}(t))$, eigenstates $|n(\mathbf{R}(t))\rangle$, and energy spectrum $E_n(\mathbf{R}(t))$, we search solutions to the time-dependent S.-Eq.,

$$i\hbar\partial_t \left|\psi_n[\mathbf{R}(t)]\right\rangle = \hat{\mathcal{H}}[\mathbf{R}(t)] \left|\psi_n[\mathbf{R}(t))]\right\rangle , \tag{16}$$

while traversing a path in $\mathbf{R}$ space 'sufficiently slow' so as to ensure that a system which starts in an eigenstate $|n\rangle$ at time $t = 0$ remains in the corresponding eigenstate at $t > 0$. According to the adiabatic theorem, there is a threshold speed for traversing any parameter path, below which this condition is met.

The full expression for the wavefunction is

$$|\psi(\mathbf{R})\rangle = \exp\left(i\gamma_n(t)\right) \exp\left(-\frac{i}{\hbar}\int_0^t dt' E_n\left[\mathbf{R}(t')\right]\right) \left|n\left[\mathbf{R}(t)\right]\right\rangle \tag{17}$$

where $\left|n\left[\mathbf{R}(t)\right]\right\rangle$ satisfies the time-independent Schroedinger equation,

$$\hat{\mathcal{H}}[\mathbf{R}(t)] \left|n\left[\mathbf{R}(t)\right]\right\rangle = E_n \left|n\left[\mathbf{R}(t)\right]\right\rangle \tag{18}$$

Besides the dynamical phase factor, an additional phase term γ_n is necessary to fulfill the S.-Eq. as we will show in the following. This phase is called geometrical phase or Berry phase.

From here, we abbreviate $\mathbf{R}(t) \equiv \mathbf{R}$ and have

$$\begin{aligned} i\hbar\partial_t \left|\psi(\mathbf{R})\right\rangle &= e^{i\gamma_n(t)} e^{-\frac{i}{\hbar}\int_0^t dt' E_n\left[\mathbf{R}(t')\right]} i\hbar \left[i\frac{\partial\gamma_n}{\partial t} - \frac{i}{\hbar}E_n\left[\mathbf{R}\right] + \frac{\partial}{\partial t}\right] \left|n(\mathbf{R})\right\rangle \\ &= e^{i\gamma_n(t)} e^{-\frac{i}{\hbar}\int_0^t dt' E_n\left[\mathbf{R}(t')\right]} \hat{\mathcal{H}}(\mathbf{R}) \left|n(\mathbf{R})\right\rangle \end{aligned} \tag{19}$$

Now to simplify, we remove the dynamical phase factor and multiply this equation by $\langle n(\mathbf{R})|$ from the left-hand side, yielding

$$\frac{\partial\gamma_n}{\partial t} = i \left\langle n(\mathbf{R})\right| \frac{\partial}{\partial t} \left|n(\mathbf{R})\right\rangle = i \left\langle n(\mathbf{R})\right| \frac{\partial}{\partial\mathbf{R}} \left|n(\mathbf{R})\right\rangle \cdot \frac{\partial\mathbf{R}}{\partial t} \tag{20}$$

and we can define the shorthand for the parameter-space velocity $\dot{\mathbf{R}} = \partial\mathbf{R}/\partial t$ (sufficiently slow to satisfy the adiabatic condition). Note that the bra/ket expression is a contraction of wavefunctions in Hilbert space, but also a 3-vector in the parameter space of $\mathbf{R}$.

As will be shown below, the expression for γ_n is not gauge-invariant in the general case; however, when considering a closed loop $\mathcal{C}$ in $\mathbf{R}$-space, a physically meaningful quantity is obtained:

$$\begin{aligned}\gamma_n[\mathcal{C}] &= \int_0^T \frac{\partial \gamma_n}{\partial t} dt = i \int_0^T dt \, \langle n(\mathbf{R})| \frac{\partial}{\partial \mathbf{R}} |n(\mathbf{R})\rangle \cdot \dot{\mathbf{R}} = i \oint_{\mathcal{C}} d\mathbf{l}_{\mathbf{R}} \cdot \langle n(\mathbf{R})| \frac{\partial}{\partial \mathbf{R}} |n(\mathbf{R})\rangle \\ &\equiv - \oint_{\mathcal{C}} d\mathbf{l}_{\mathbf{R}} \cdot \mathbf{A}_n(\mathbf{R}) \end{aligned} \tag{21}$$

where the Berry vector potential, Berry connection, or emergent vector potential was introduced:

$$\mathbf{A}_n = -i \langle n(\mathbf{R})| \frac{\partial}{\partial \mathbf{R}} |n(\mathbf{R})\rangle \tag{22}$$

This is a vector in the $\mathbf{R}$-space. Further in the 3D case, the Berry curvature or emergent magnetic field is defined as $\mathbf{\Omega}_n(\mathbf{R}) = \nabla_{\mathbf{R}} \times \mathbf{A}_n$ so that

$$\gamma_n = - \int d^2\mathbf{R} \, \mathbf{\Omega}_n(\mathbf{R}) \cdot \hat{n}_{\mathbf{R}} \tag{23}$$

where $\hat{n}_{\mathbf{R}}$ is a normal vector on the surface of integration in $\mathbf{R}$ space.

3.2 Berry curvature (not Berry phase) is gauge invariant

Consider a gauge transformation of the eigenstates

$$|n(\mathbf{R})\rangle \to \exp(i\Lambda(\mathbf{R})) \; |n(\mathbf{R})\rangle \tag{24}$$

The associated change of the Berry connection is

$$\begin{aligned}\mathbf{A}'_n &= -i \langle n(\mathbf{R})| e^{-i\Lambda} \frac{\partial}{\partial \mathbf{R}} e^{i\Lambda} |n(\mathbf{R})\rangle \\ &= -i \langle n| \frac{\partial}{\partial \mathbf{R}} |n\rangle - i \langle n| i \frac{\partial \Lambda}{\partial \mathbf{R}} |n\rangle = \mathbf{A}_n + \frac{\partial \Lambda}{\partial R} \end{aligned} \tag{25}$$

i.e. mathematically equivalent to a gauge transformation of the electromagnetic field with a spatially dependent scalar function $\Lambda(\mathbf{x})$. The Berry curvature $\mathbf{\Omega}_n$ remains gauge invariant.

Gauge transformation and Berry phase:

We further illustrate how the Berry *phase* is gauge invariant only modulo 2π (in contrast to $\mathbf{\Omega}_n$, a locally gauge invariant object). As deduced above, we have $\gamma_n = -\oint_{\mathcal{C}} d\mathbf{l}_{\mathbf{R}} \cdot \mathbf{A}_n(\mathbf{R})$. Under gauge transformation,

$\gamma_n' = \gamma_n - \oint_{\mathcal{C}} d\mathbf{l}_{\mathbf{R}} \cdot (\nabla_{\mathbf{R}}\Lambda)$ and $\Lambda(t=T) - \Lambda(t=0) = 2\pi N$ where $N \in \mathbb{Z}$ is required to ensure that the wave function is single-valued at a fixed point in $\mathbf{R}$-space.

3.3 Gauge transformation and singularities in parameter space

Unlike the case of a 'real' magnetic field, there can be sinks and sources (monopoles) of $\mathbf{\Omega}_n(\mathbf{r})$:

$$\nabla_{\mathbf{R}} \cdot \mathbf{\Omega}_n(\mathbf{R}) \neq 0 \tag{26}$$

due to emergent monopoles in parameters space. In this case, no single vector potential $\mathbf{A}_n(\mathbf{R})$ exists which could satisfy $\nabla_{\mathbf{R}} \times \mathbf{A}_n(\mathbf{R}) = \mathbf{\Omega}_n(\mathbf{R})$ everywhere in parameter space, without becoming singular at some points.

Comment 1: Stokes' theorem for $\mathbf{A}_n(\mathbf{R})$,

$$\oint_{\mathcal{C}} d\mathbf{l}_{\mathbf{R}} \cdot \mathbf{A}_n(\mathbf{R}) = \int_{\mathcal{S}(\mathcal{C})} d^2\mathbf{R}\,(\nabla_{\mathbf{R}} \times \mathbf{A}_n(\mathbf{R})) \cdot \hat{n}_{\mathbf{R}} \tag{27}$$

is only valid if the loop and surface are in a region of $\mathbf{R}$-space where $\mathbf{\Omega}_n(\mathbf{R}) = \nabla_{\mathbf{R}} \times \mathbf{A}_n(\mathbf{R})$ has no singularities.

Comment 2: "Moving $\mathcal{S}(\mathcal{C})$ across a monopole"; we start with a loop $\mathcal{C}$ and the narrow surface (neck) $\mathcal{S}_1(\mathcal{C})$ that is the tightest surface surrounded by the loop. Let a monopole be situated above this surface. We let the surface expand upwards; then we let the surface "catch" the monopole and finally envelop the monopole.

Recall the previous problem of an electron in the environment of a magnetic monopole, with two vector potentials $\mathbf{A}^{\mathrm{S}}$ and $\mathbf{A}^{\mathrm{N}}$. In this language, we can use $\mathbf{A}^{\mathrm{S}}$ for $\mathcal{S}_1(\mathcal{C})$, but should use $\mathbf{A}^{\mathrm{N}}$ for $\mathcal{S}_2(\mathcal{C})$; this is because the former surface is located entirely below the monopole, and the latter surface does not include the south pole.

In summary: When calculating the Berry phase, the correct gauge for the vector potential depends on the surface of integration. What is more, the gauge transformation for going from $\mathbf{A}^{\mathrm{S}}$ to $\mathbf{A}^{\mathrm{N}}$ changes $\gamma_n(\mathcal{C})$

by $2\pi N$, with $N \in \mathcal{Z}$. The gauge degree of freedom corresponds to a choice of surface of integration in the Berry phase formula.

Comment:

1. The emergent magnetic field $\mathbf{\Omega}_n(\mathbf{R})$ is a *local* quantity, whereas γ_n is only properly defined on a loop.

2. $\mathbf{\Omega}_n(\mathbf{R})$ is gauge invariant and physically observable, whereas γ_n is affected by a gauge transformation and not directly observable (only modulo 2π).